Hold on to your homework (we will turn in at the end of class)

3.18c $\int_{S^2} V\cdot N\, dA$, $N =$ inward unit normal

$V = (x, xy, 1)$

$= -\int_{S^2} V\cdot n\, dA$

n ← outward normal

S^2 ← unit sphere in $\mathbb{R}^3$

$\phi(u,v) = (\sin(u)\cos(v), \sin(u)\sin(v), \cos(u))$

u ← ϕ, v ← θ, $\rho = 1$

$du \wedge dv$ is outward

$\phi_u = (\cos(u)\cos(v), \cos(u)\sin(v), -\sin(u))$

$\phi_v = (-\sin(u)\sin(v), \sin(u)\cos(v), 0)$

$$\phi_u \times \phi_v = \begin{vmatrix} i & j & k \\ \cos(u)\cos(v) & \cos(u)\sin(v) & -\sin(u) \\ -\sin(u)\sin(v) & \sin(u)\cos(v) & 0 \end{vmatrix}$$

$$= i\left(\sin^2(u)\cos(v)\right) - j\left(-\sin^2(u)\sin(v)\right) + k\left(\sin(u)\cos(u)\right)$$

$$V(\phi(u,v))\quad \phi_u \times \phi_v = \left(\sin^2(u)\cos(v), \sin^2(u)\sin(v), \sin(u)\cos(u)\right)$$

$$V = \left(\sin(u)\cos(v), \sin^2(u)\sin(v)\cos(v), 1\right)$$

$$-\int_{v=0}^{2\pi}\int_{u=0}^{\pi}\left(\sin^3(u)\cos^2(v) + \sin^4(u)\sin^2(v) + \sin(u)\cos(u)\right) du\, dv$$

θ $\quad$ ϕ

rectangular $\phi_1(u,v) = (u, v, \sqrt{1-u^2-v^2})$ top half

$x^2+y^2+z^2=1$

$du \wedge dv$ is outward. ✓

$z = \pm\sqrt{1-x^2-y^2}$

top half:

$\phi_{1u} = \left(1, 0, \frac{1}{2}(1-u^2-v^2)^{-1/2}(-2u)\right)$

$\phi_{1v} = \left(0, 1, \frac{1}{2}(1-u^2-v^2)^{-1/2}(-2v)\right)$

$(\phi_{1u} \times \phi_{1v}) = \quad i \quad j \quad k$

$$\underbrace{-\underbrace{\int_{u=-1}^{1}\int_{v=-\sqrt{1-u^2}}^{\sqrt{1-u^2}} V\cdot(\phi_u\times\phi_v)\,dv\,du}_{\text{outward flux}}}_{\text{inward flux}}$$

bottom half

$\phi_2 = (u, v, -\sqrt{1-u^2-v^2})$

$du \wedge dv$ — upward (inside normal)

$$\text{bottom} + \int_{u=-1}^{1}\int_{v=-\sqrt{1-u^2}}^{\sqrt{1-u^2}} V\cdot(\phi_u\times\phi_v)\,dv\,du$$

---

differential

$u = \phi$

$v = \theta$

$x = \sin(u)\cos(v)$

$y = \sin(u)\sin(v)$

$z = \cos(u)$

$V = (x, xy, 1)$

Outward flux $= \iint x\cdot(dy \wedge dz) + xy\,(dz \wedge dx) + 1\,(dx \wedge dy)$

inward flux $=$ $-$ outward flux.

$$-\iint V\cdot \underset{\substack{\uparrow\\ \text{outward}\\ \text{normal}}}{n}\, dA = \underset{\substack{\text{unit}\\ \text{ball}}}{\iiint}(\operatorname{div} V)\, d\mathrm{vol}$$

$V=(x,xy,1)$

$$\operatorname{div} V = (1+x+0) = 1+x$$

$$= -\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=0}^{1}\overbrace{(1+\rho\sin\phi\cos\theta)}^{\operatorname{div} V}\cdot\overbrace{\rho^2\sin\phi\, d\rho\, d\phi\, d\theta}^{\text{volume form}}$$

General Stokes Thm

$$\int_{\partial \text{Bubba}} \underset{\text{differential form}}{\alpha} = \int_{\substack{\text{Bubba}\\ \text{inside}}} d\alpha$$

If
$$\begin{cases}\alpha = V\cdot ds & \text{1-form}\\ d\alpha = (\operatorname{curl} V)\cdot \underset{\substack{\uparrow\\ \text{given by RHR}}}{n}\, dA & \text{– 2form}\end{cases}$$

If
$$\begin{cases}\alpha = W\cdot n\, dA & \text{2-form}\\ d\alpha = (\operatorname{div} W)\cdot\begin{pmatrix}\text{volume}\\ \text{form}\end{pmatrix} & \text{3-form}\end{cases}$$

$$\Rightarrow \int_{\substack{\text{closed}\\ \text{curve}}} V\cdot ds = \int_{\substack{\text{inside}\\ \text{surface}}} (\operatorname{curl} V)\cdot n\, dA \quad \leftarrow \text{match curve direction by RHR}$$

$$\Rightarrow \int_{\substack{\text{closed}\\ \text{surface}}} W\cdot \underset{\uparrow\,\text{outward}}{n}\, dA = \int_{\substack{\text{inside}\\ \text{volume}}} (\operatorname{div} W)\,(\underset{\text{volume form}}{dV})$$

3.22 Use Div. Thm to rewrite $B = (z+xy, -3, x)$

$\int_S B \cdot N \, dA$ ↑ outward normal

S = boundary of region between $z = x^2+y^2$ & $z = 2x+y$

$$= \int_{\text{inside}} (\operatorname{div} B) \begin{pmatrix}\text{volume}\\ \text{form}\end{pmatrix}$$

$$(\operatorname{div} B) = \partial_x(z+xy) + \partial_y(-3) + \partial_z(x)$$
$$= y$$

$$x^2+y^2 \le z \le 2x+y$$

intersection of surfaces: $z = x^2+y^2 = 2x+y$

$$x^2 - 2x \underset{+1}{} + y^2 - y \underset{+\frac{1}{4}}{} = 0 \underset{+1+\frac{1}{4}}{}$$

$$(x-1)^2 + (y-\tfrac{1}{2})^2 = \frac{5}{4} = \left(\frac{\sqrt{5}}{2}\right)^2$$

↓ solve for y $(y-\frac{1}{2})^2 = \frac{5}{4} - (x-1)^2$

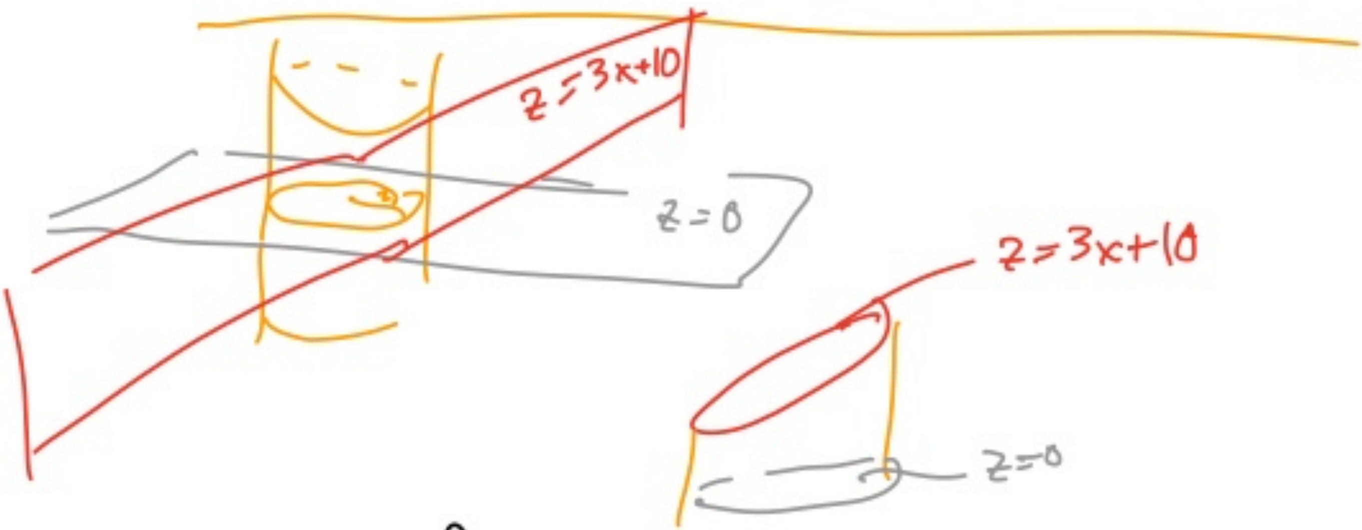
$$\int_{-\frac{\sqrt{5}}{2}}^{\frac{\sqrt{5}}{2}} \int_{\frac{1}{2}-\sqrt{}}^{\frac{1}{2}+\sqrt{}} \int_{z=x^2+y^2}^{2x+y} y \, dz\, dy\, dx$$

To convert to cylindrical:

$$\left.\begin{aligned}\tilde{x} &= x-1\\ \tilde{y} &= y-\tfrac{1}{2}\end{aligned}\right) \quad \begin{aligned} d\tilde{x} &= dx \\ d\tilde{y} &= dy\end{aligned} \qquad y = \left(\tfrac{1}{2} + \tilde{y}\right)$$

$$\left.\begin{aligned}\tilde{x} &= r\cos\theta = x-1\\ \tilde{y} &= r\sin\theta = y-\tfrac{1}{2}\end{aligned}\right) \quad \begin{aligned} &dx\,dy \\ &= r\,dr\,d\theta\end{aligned}$$

Example Find the outward flux of $F=(x^2,y^2,z^2)$ over the surface that is the boundary of the set $B=\{(x,y,z): x^2+y^2\le 4 \text{ and } 0\le z\le 3x+10\}$



$$= \iiint_{\text{inside}} (\operatorname{div} F)\,(\text{volume form})$$

$F=(x^2,y^2,z^2)$

$\operatorname{div} F = 2x+2y+2z$

Cylindrical

$0\le r\le 2$

$-\pi\le\theta\le\pi$

$0\le z\le 3x+10 = 3r\cos\theta+10$

$2r\cos\theta + 2r\sin\theta + 2z$

$$\int_{\theta=-\pi}^{\pi}\int_{r=0}^{2}\int_{z=0}^{3r\cos\theta+10}(2r\cos\theta+2r\sin\theta+2z)\,r\,dz\,dr\,d\theta$$

$2r^2\cos\theta + 2r^2\sin\theta + 2rz$

$$= \int_{\theta=-\pi}^{\pi}\int_{r=0}^{2} 2r^2\cos\theta\, z + 2r^2\sin\theta\, z + rz^2 \Big|_0^{3r\cos\theta+10}$$

$$= \int_{\theta=-\pi}^{\pi} \int_{r=0}^{2} 2r^2\cos\theta\,(3r\cos\theta+10) + 2r^2\sin\theta\,(3r\cos\theta+10) + r\underbrace{(3r\cos\theta+10)^2}_{9r^2\cos^2\theta + 60 r\cos\theta + 100} \, dr\, d\theta$$

$$= \int_{\theta=-\pi}^{\pi} \int_{r=0}^{2} \Big(6r^3\underbrace{\cos^2\theta}_{\pi} + \underbrace{20r^2\cos\theta}_{0} + \underbrace{6r^3\sin\theta\cos\theta}_{0} + \underbrace{20r^2\sin\theta}_{0} + 9r^3\underbrace{\cos^2\theta}_{\pi} + \underbrace{60r^2\cos\theta}_{0} + \underbrace{100r}_{2\pi} \Big) dr\, d\theta$$

fubinate
$$= \int_{r=0}^{2} \int_{\theta=-\pi}^{\pi} \quad '' \quad d\theta\, dr$$

$$= \int_{r=0}^{2} (\underbrace{6\pi r^3 + 9\pi r^3}_{15\pi r^3} + 200\pi r)\, dr$$

$$= \frac{15\pi}{4} r^4 + 100\pi r^2 \Big|_0^2$$

$$= \frac{15\pi}{4}\cdot 16 + 100\pi \cdot 4 = 60\pi + 400\pi$$

$$= \boxed{460\pi}.$$

Interesting Facts about curl, div, grad
$\nabla\times$, $\nabla\cdot$, ∇

① $\operatorname{div}(\operatorname{curl} V) = 0$ for any vector field V.

$$'' \; d(d(\text{1-form})) = 0 \qquad d(d\omega) = 0$$

$$\operatorname{div}(\partial_y V_3 - \partial_z V_2,\ \partial_z V_1 - \partial_x V_3,\ \partial_x V_2 - \partial_y V_1) \qquad \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= (\cancel{\partial_x\partial_y V_3} - \cancel{\partial_x\partial_z V_2} + \cancel{\partial_y\partial_z V_1} - \cancel{\partial_y\partial_x V_3} + \cancel{\partial_z\partial_x V_2} - \cancel{\partial_z\partial_y V_1}) = 0$$

(2) $\text{curl}(\text{grad } f) = 0$ for any function f

$\nabla \times \nabla f$

$\parallel$

$d(df) = 0$

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \partial_x f & \partial_y f & \partial_z f \end{vmatrix} = (0, 0, 0)$$

again because mixed partials commute.

(3) $\text{div}(\text{grad } f) = \nabla \cdot \nabla f$ (not always zero)!

$$= \nabla \cdot (\partial_x f, \partial_y f, \partial_z f, \dots)$$

$$= \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} + \cdots \right) = \Delta f$$

Laplacian of f.

very important operator

$$\frac{\partial f}{\partial t} - \Delta f = 0$$

Heat equation

Diffusion equation